Fourier series and transform

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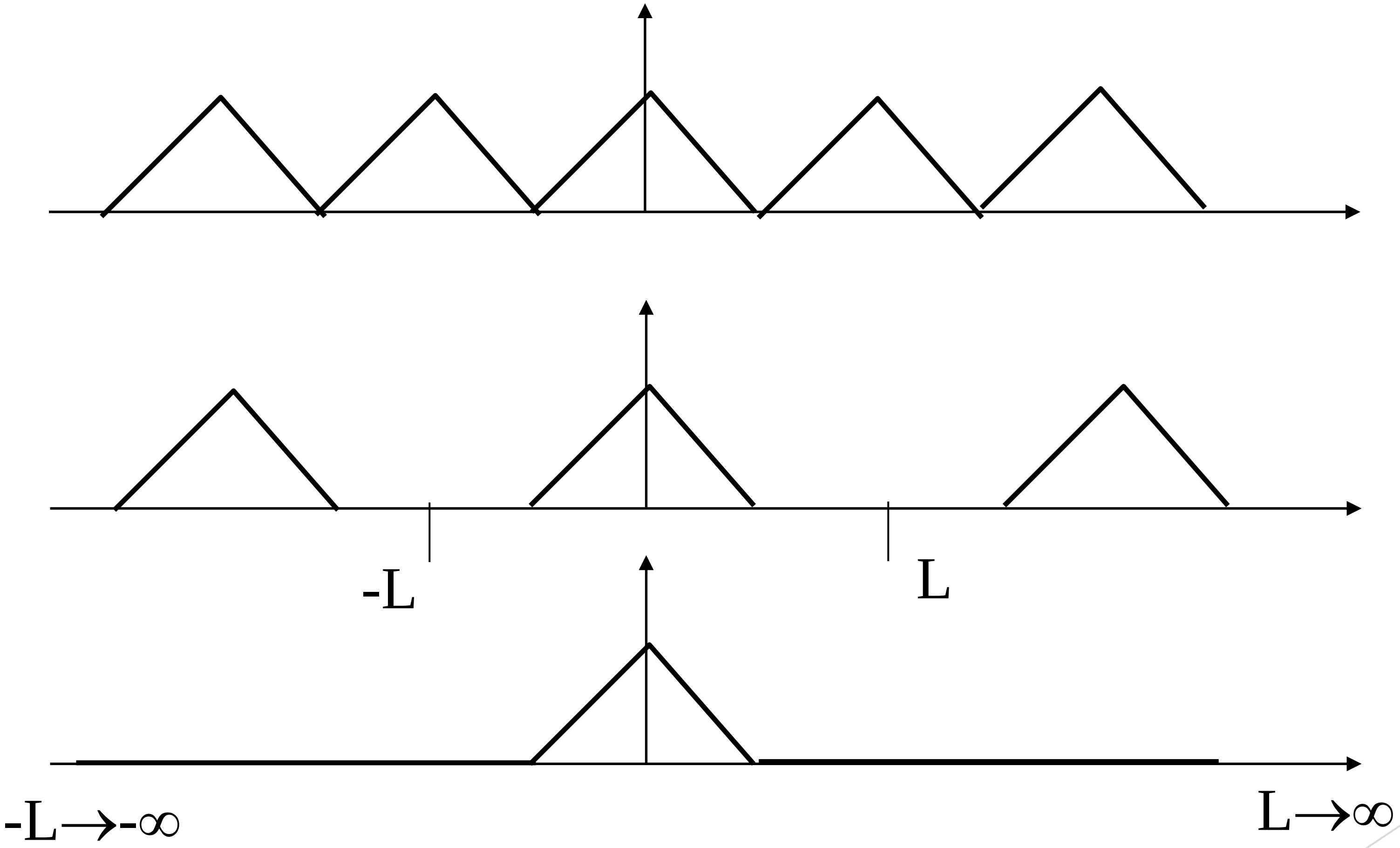
Fourier Integrals

- For non-periodic applications (or a specialized Fourier series when the period of the function is infinite: L→∞)



$$f_L(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(w_n x) + b_n \sin(w_n x)\right), w_n = \frac{n\pi}{L}$$

$$= \sum_{n=0}^{\infty}\left(a_n \cos(w_n x) + b_n \sin(w_n x)\right)$$

Note that : $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

$$f_L(x) =$$

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\left[\cos(w_n x)\Delta w\int_{-L}^{L} f_L(v)\cos(w_n v)dv + \sin(w_n x)\Delta w\int_{-L}^{L} f_L(v)\sin(w_n v)dv\right]$$

As $\mathrm{L} \to \infty, \Delta \mathrm{w} \to 0, \Rightarrow \sum(\,)\Delta w \to \int(\,)dw$

$$f_L(x) =$$

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\left[\cos(w_n x)\Delta w\int_{-L}^{L} f_L(v)\cos(w_n v)dv + \sin(w_n x)\Delta w\int_{-L}^{L} f_L(v)\sin(w_n v)dv\right]$$

$$= \frac{1}{\pi}\int_0^{\infty}\left[\cos(wx)\int_{-\infty}^{\infty} f(v)\cos(wv)dv + \sin(wx)\int_{-\infty}^{\infty} f(v)\sin(wv)dv\right]dw$$

$$= \int_0^{\infty}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)dv[\cos(wx)] + \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)dv[\sin(wx)]\right\}dw$$

$f(x) = \int_0^{\infty}[A(w)\cos(wx) + B(w)\sin(wx)]dw$: **Fourier integral of f(x)**

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)dv, B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)dv$

Fourier Cosine & Sine Integrals

$$\text{If the function f(x) is even} \Rightarrow \mathrm{A(w)} = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(wv)dv$$

$$= \frac{1}{\pi}\int_{-\infty}^{0}(\,)dv + \frac{1}{\pi}\int_{0}^{\infty}(\,)dv = \frac{2}{\pi}\int_{0}^{\infty} f(v)\cos(wv)dv$$

$$B(w) = 0$$

$$f(x) = \int_{0}^{\infty} A(w)\cos(wx)dw : \text{Fourier Cosine Integral}$$

$$\text{If the function f(x) is odd} \Rightarrow B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(wv)dv$$

$$A(w) = 0,\; f(x) = \int_{0}^{\infty} B(w)\sin(wx)dw : \text{ Fourier Sine Integral}$$

Fourier Cosine Transform

For an even function f(x) :

$$f(x)=\int_0^\infty A(w)\cos(wx)dw, \quad \text{where } A(w)=\frac{2}{\pi}\int_0^\infty f(v)\cos(wv)dv.$$

Define $A(w)=\sqrt{\frac{2}{\pi}}\,\hat{f}_c(w)$

$$\hat{f}_c(w)=\sqrt{\frac{\pi}{2}}A(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(wx)dx, \quad v \text{ has been replaced by x}$$

$\hat{f}_c(w)$ is called the Fourier cosine transform of f(x)

$$f(x)=\int_0^\infty A(w)\cos(wx)dw=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos(wx)dw$$

$f(x)$ is the inverse Fourier cosine transform of $\hat{f}_c(w)$

Fourier Sine Transform

Similarly, for an odd function f(x) :

$$f(x)=\int_0^\infty B(w)\sin(wx)\,dw,\quad \text{where } B(w)=\frac{2}{\pi}\int_0^\infty f(v)\sin(wv)\,dv.$$

Define $B(w)=\sqrt{\frac{2}{\pi}}\,\hat{f}_S(w)$

$$\hat{f}_S(w)=\sqrt{\frac{\pi}{2}}B(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(wx)\,dx,\quad v \text{ has been replaced by x}$$

$\hat{f}_S(w)$ is called the Fourier sine transform of f(x)

$$f(x)=\int_0^\infty B(w)\sin(wx)\,dw=\sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_S(w)\sin(wx)\,dw$$

$f(x)$ is the inverse Fourier sine transform of $\hat{f}_S(w)$

THANK YOU