RECURRENCE FORMULAE OF BESSELS FUNCTION

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RECURRENCE FORMULAE :-

(1) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$ **or** $\int x^n J_{n-1}(x)dx = x^n J_n(x)$

Proof: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$\Rightarrow x^n J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2n+2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$

$\Rightarrow \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty}(-1)^r \frac{2(n+r)x^{2n+2r-1}}{2^{n+2r}} \frac{1}{r!\,(n+r)\Gamma(n+r)}$

$\because \Gamma(n+r+1) = (n+r)\Gamma(n+r)$

$= x^n \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{r!\,\Gamma((n-1)+r+1)}$

$= x^n J_{n-1}(x)$

(2) $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$ **or** $\int x^{-n}J_{n+1}(x)\,dx = -\frac{d}{dx}[x^{-n}J_n(x)]$

Proof: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\,\Gamma(n+r+1)}$

$$\Rightarrow x^{-n}J_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{x^{2r}}{2^{n+2r}}\frac{1}{r!\,\Gamma(n+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^{-n}J_n(x)] = \sum_{r=1}^{\infty}(-1)^r\frac{2r\,x^{2r-1}}{2^{n+2r}}\frac{1}{(r-1)!\;r\,\Gamma(n+r+1)}$$

$$= x^{-n}\sum_{r=1}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r-1}\frac{1}{(r-1)!\;\Gamma(n+r+1)}$$

$$= -x^{-n}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{(n+1)+2k}\frac{1}{k!\;\Gamma((n+1)+k+1)}$$

Putting $r = k+1$

$$= -x^{-n}J_{n+1}(x)$$

(3) $\boldsymbol{J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)}$

Proof: From recurrence relation (1)

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\Rightarrow x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

Dividing by x^n, we get

$$J_n'(x) + \frac{n}{x} J_n(x) = J_{n-1}(x)$$

$$\Rightarrow J_n'(x) = J_{n-1}(x) - \frac{n}{x} J_n(x)$$

(4) $\boldsymbol{J_n'(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x)}$

Proof: From recurrence relation (2)

$$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$$\Rightarrow x^{-n} J_n'(x) - n x^{-n-1} J_n(x) = -x^{-n} J_{n+1}(x)$$

Dividing by x^{-n}, we get

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x} J_n(x)$$

(5) $$\boldsymbol{J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]}$$

Proof: Adding recurrence relations (3) and (4), we get

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

(6) $$\boldsymbol{2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]}$$

Proof: Subtracting recurrence relations (3) from (4), we get

$$2\frac{n}{x} J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\Rightarrow 2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$