Vector Spaces

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I.1. Definition and Examples

Definition : (Real) Vector Space (V, ♣ ; $\mathbb{R}$)

A vector space (over $\mathbb{R}$) consists of a set V along with 2 operations '♣' and '♦' s.t.

(1) For the vector addition ♣ :

$\forall$ $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u} \in V$

a) $\boldsymbol{v}$ ♣ $\boldsymbol{w} \in V$ (Closure)

b) $\boldsymbol{v}$ ♣ $\mathbf{w} = \boldsymbol{w}$ ♣ $\mathbf{v}$ (Commutativity)

c) ($\boldsymbol{v}$ ♣ $\boldsymbol{w}$) ♣ $\boldsymbol{u} = \boldsymbol{v}$ ♣ ($\boldsymbol{w}$ ♣ $\boldsymbol{u}$) (Associativity)

d) $\exists\ \mathbf{0} \in V$ s.t. $\boldsymbol{v}$ ♣ $\mathbf{0} = \boldsymbol{v}$ (Zero element)

e) $\exists\ -\boldsymbol{v} \in V$ s.t. $\boldsymbol{v}$ ♣ $(-\boldsymbol{v}) = \mathbf{0}$ (Inverse)

(2) For the scalar multiplication ♦ :

$\forall$ $\boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in \mathbb{R}$, [$\mathbb{R}$ is the real number field $(\mathbb{R},+,\times)$

f) a ♦ $\boldsymbol{v} \in V$ (Closure)

g) $(a + b)$ ♦ $\boldsymbol{v}$ = (a ♦ $\boldsymbol{v}$) ♣ (b ♦ $\boldsymbol{v}$) (Distributivity)

h) a ♦ ($\boldsymbol{v}$ ♣ $\boldsymbol{w}$) = (a ♦ $\boldsymbol{v}$) ♣ (a ♦ $\boldsymbol{w}$)

i) $(a \times b)$ ♦ $\boldsymbol{v}$ = a ♦ (b ♦ $\boldsymbol{v}$) (Associativity)

j) 1 ♦ $\boldsymbol{v} = \boldsymbol{v}$

- ♣ is always written as + so that one writes $\boldsymbol{v} + \boldsymbol{w}$ instead of $\boldsymbol{v}$ ♣ $\boldsymbol{w}$
- $\times$ and ♦ are often omitted so that one writes $a\, b\, \boldsymbol{v}$ instead of $(a \times b)$ ♦ $\boldsymbol{v}$

Definition in Conventional Notations

Definition : (Real) Vector Space $(V, + ; \mathbb{R})$

A vector space (over $\mathbb{R}$) consists of a set V along with 2 operations '+' and ' ' s.t.

(1) For the vector addition + :

$\forall$ $\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u} \in V$

a) $\boldsymbol{v} + \boldsymbol{w} \in V$ (Closure)

b) $\boldsymbol{v} + \mathbf{w} = \boldsymbol{w} + \mathbf{v}$ (Commutativity)

c) $(\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{u} = \boldsymbol{v} + (\boldsymbol{w} + \boldsymbol{u})$ (Associativity)

d) $\exists\, \mathbf{0} \in V$ s.t. $\boldsymbol{v} + \mathbf{0} = \boldsymbol{v}$ (Zero element)

e) $\exists\, -\boldsymbol{v} \in V$ s.t. $\boldsymbol{v} - \boldsymbol{v} = \mathbf{0}$ (Inverse)

(2) For the scalar multiplication :

$\forall$ $\boldsymbol{v}, \boldsymbol{w} \in V$ and $a, b \in \mathbb{R}$, [$\mathbb{R}$ is the real number field $(\mathbb{R},+,\times)$]

f) $a\, \boldsymbol{v} \in V$ (Closure)

g) $(a + b)\, \boldsymbol{v} = a\, \boldsymbol{v} + b\, \boldsymbol{v}$ (Distributivity)

h) $a\, (\boldsymbol{v} + \boldsymbol{w}) = a\, \boldsymbol{v} + a\, \boldsymbol{w}$

i) $(a \times b)\, \boldsymbol{v} = a\, (b\, \boldsymbol{v}) = a\, b\, \boldsymbol{v}$ (Associativity)

j) $1\, \boldsymbol{v} = \boldsymbol{v}$

Example 1: Solution Space of a Linear Homogeneous Differential Equation

$$S = \left\{ f : \mathbb{R} \to \mathbb{R} \;\middle|\; \frac{d^2 f}{d x^2} + f = 0 \right\}$$ is a vector space with

Vector addition: $\left(f + g\right)\left(x\right) \equiv f\left(x\right) + g\left(x\right)$

Scalar multiplication: $\left(a f\right)\left(x\right) \equiv a f\left(x\right)$ $\quad a \in \mathbb{R}$

Zero element: $zero(x) = 0$

Inverse: $\left(-f\right)\left(x\right) \equiv -f\left(x\right)$

Closure: $\frac{d^2 f}{d x^2} + f = 0 \;\&\; \frac{d^2 g}{d x^2} + g = 0 \;\rightarrow\; \frac{d^2\left(a f + b g\right)}{d x^2} + \left(a f + b g\right) = 0$

Example 2: Solution Space of a System of Linear Homogeneous Equations

Remarks:

- Definition of a mathematical structure is not unique.
- The accepted version is time-tested to be most concise & elegant.

Lose Ends

In any vector space V,

1. $0\,\boldsymbol{v} = \mathbf{0}$.
2. $(-1)\,\boldsymbol{v} + \boldsymbol{v} = \mathbf{0}$.
3. $a\,\mathbf{0} = \mathbf{0}$.

$\forall\ \boldsymbol{v} \in V$ and $a \in \mathbb{R}$.

Proof:

1. $\mathbf{0} = \mathbf{v} - \mathbf{v} = (1+0)\,\mathbf{v} - \mathbf{v} \quad = \mathbf{v} + 0\mathbf{v} - \mathbf{v} \quad = 0\mathbf{v}$
2. $(-1)\,\mathbf{v} + \mathbf{v} = (-1+1)\,\mathbf{v} \quad = 0\mathbf{v} \quad = \mathbf{0}$
3. $a\,\mathbf{0} = a\,(0\ \mathbf{v}) \quad = (a\,0)\,\mathbf{v} \quad = 0\,\mathbf{v} \quad = \mathbf{0}$

Exercise

1. At this point "the same" is only an intuition, but nonetheless for each vector space identify the k for which the space is "the same" as $\mathbb{R}^k$.

(a) The 2×3 matrices under the usual operations

(b) The $n \times m$ matrices (under their usual operations)

(c) This set of 2×2 matrices

$$\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| \, a+b+c=0 \right\}$$

2.

(a) Prove that every point, line, or plane thru the origin in $\mathbb{R}^3$ is a vector space under the inherited operations.

(b) What if it doesn't contain the origin?

THANK YOU