

A GENERATING FUNCTION OF LEGENDRE'S POLYNOMIAL

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Prove that $P_n(x)$ is the coefficient of z^n in the expansion of $(1 - 2xz + z^2)^{-1/2}$ in ascending powers of x .

Proof. $(1 - 2xz + z^2)^{-1/2} = [1 - z(2x - z)]^{-1/2}$

Expanding R.H.S. by Binomial Theorem, we have

$$\begin{aligned} (1 - 2xz + z^2)^{-1/2} &= 1 + \frac{1}{2}z(2x - z) + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}z^2(2x - z)^2 + \dots \\ &\quad + \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{1}{2} - n + 1\right)}{n!}(-z)^n(2x - z)^n + \dots \dots (1) \end{aligned}$$

Now coefficient of z^n in $(n + 1)$ th term *i.e.* $\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{1}{2} - n + 1\right)}{n!}(-z)^n(2x - z)^n$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{1}{2} - n + 1\right)}{n!}(-1)^n(2x)^n$$

$$= \frac{1.3.5\dots(2n-1)}{2^n \cdot n!} (2)^n \cdot x^n = \frac{1.3.5\dots(2n-1)}{n!} x^n \quad \dots (2)$$

Coefficient of z^n in n th term i.e. $\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{1}{2}-n+2\right)}{(n-1)!} (-z)^{n-1} (2x-z)^{n-1}$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{1}{2}-n+2\right)}{(n-1)!} (-1)^{n-1} [-(n-1) (2x)^{n-2}]$$

$$= \frac{1.3.5\dots(2n-3)}{2^{n-1} \cdot (n-1)!} (2)^{n-2} (n-1)x^{n-2} = \frac{1.3.5\dots(2n-3)}{2 \cdot (n-1)!} (n-1)x^{n-2}$$

$$= \frac{1.3.5\dots(2n-3)}{2 \cdot (n-1)!} \times \frac{(2n-1)}{(2n-1)} (n-1)x^{n-2} = \frac{1.3.5\dots(2n-3)(2n-1)}{n!} \times \frac{n(n-1)}{2(2n-1)} x^{n-2}$$

... (3)

Coefficient of x^n in

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{1}{2}-n+3\right)}{(n-2)!} z^{n-2} (2x-z)^{(n-2)}$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\dots\left(-\frac{1}{2}-n+3\right)}{(n-2)!} \times (-1)^{n-2} \times \frac{(n-2)(n-3)}{2!} (2x)^{n-4}$$

$$= \frac{1.3.5\dots(2n-5)}{2^{n-2} (n-2)!} \times \frac{(n-2)(n-3)}{2!} (2x)^{n-4}$$

$$\begin{aligned}
&= \frac{1.3.5\dots(2n-5)(2n-3)(2n-1)}{4(n-2)!} \times \frac{(n-2)(n-3)}{2(2n-3)(2n-1)} x^{n-4} \\
&= \frac{1.3.5\dots(2n-1)}{4n(n-1)(n-2)!} \times \frac{n(n-1)(n-2)(n-3)}{2(2n-3)(2n-1)} x^{n-4} \\
&= \frac{1.3.5\dots(2n-1)}{n!} \times \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} \quad \dots (4)
\end{aligned}$$

and so on.

Thus coefficient of z^n in the expansion of (1) is sum of (2), (3) and (4) etc.

$$= \frac{1.3.5\dots(2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} - \dots \right] = P_n(x)$$

Thus coefficients of $z, z^2, z^3 \dots$ etc. in (1) are $P_1(x), P_2(x), P_3(x) \dots$

Hence

$$(1 - 2xz + z^2)^{-1/2} = P_0(x) + zP_1(x) + z^2P_2(x) + z^3P_3(x) + \dots + z^n P_n(x) + \dots$$

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{n=\infty} P_n(x) \cdot z^n.$$

The background features abstract, overlapping geometric shapes in various shades of green, ranging from light lime to dark forest green. These shapes are primarily located on the left and right sides of the frame, leaving a large white central area. The shapes are composed of triangles and polygons, some with thin white outlines.

THANK YOU