A GENERATING FUCNTION OF LEGENDRE'S POLYNOMIAL

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Prove that $P_n(x)$ is the coefficient of z^n in the expansion of $(1-2xz+z^2)^{-1/2}$ in ascending powers of x.

Proof. $(1-2xz+z^2)^{-1/2} = [1-z(2x-z)]^{-1/2}$

Expanding R.H.S. by Binomial Theorem, we have

$$(1-2xz+z^2)^{\frac{-1}{2}} = 1+\frac{1}{2}z(2x-z)+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)}{2!}z^2(2x-z)^2+\ldots$$

$$+\frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}(-z)^n(2x-z)^n+\ldots \quad \ldots (1)$$

Now coefficient of z^n in $(n+1)$ th term *i.e.* $\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}(-z)^n(2x-z)^n$

$$=\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!}(-1)^n(2x)^n$$

$$= \frac{1.3.5\ldots(2n-1)}{2^n \cdot n!}(2)^n \cdot x^n = \frac{1.3.5\ldots(2n-1)}{n!}x^n \qquad \ldots (2)$$

Coefficient of z^n in nth term *i.e.* $\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+2\right)}{(n-1)!}(-z)^{n-1}(2x-z)^{n-1}$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+2\right)}{(n-1)!}(-1)^{n-1}[-(n-1)\ (2x)^{n-2}]$$

$$= \frac{1.3.5\ldots(2n-3)}{2^{n-1}\cdot(n-1)!}(2)^{n-2}(n-1)x^{n-2} = \frac{1.3.5\ldots(2n-3)}{2\cdot(n-1)!}(n-1)x^{n-2}$$

$$= \frac{1.3.5\ldots(2n-3)}{2\cdot(n-1)!}\times\frac{(2n-1)}{(2n-1)}(n-1)x^{n-2} = \frac{1.3.5\ldots(2n-3)(2n-1)}{n!}\times\frac{n\,(n-1)}{2\,(2n-1)}x^{n-2} \qquad \ldots (3)$$

Coefficient of x^n in

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+3\right)}{(n-2)!}z^{n-2}(2x-z)^{(n-2)}$$

$$= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\ldots\left(-\frac{1}{2}-n+3\right)}{(n-2)!}\times(-1)^{n-2}\times\frac{(n-2)(n-3)}{2!}(2x)^{n-4}$$

$$= \frac{1.3.5\ldots(2n-5)}{2^{n-2}(n-2)!}\times\frac{(n-2)\,(n-3)}{2!}(2x)^{n-4}$$

$$= \frac{1.3.5\ldots(2n-5)(2n-3)(2n-1)}{4(n-2)!} \times \frac{(n-2)(n-3)}{2(2n-3)(2n-1)} x^{n-4}$$

$$= \frac{1.3.5\ldots(2n-1)}{4n(n-1)(n-2)!} \times \frac{n(n-1)(n-2)(n-3)}{2(2n-3)(2n-1)} x^{n-4}$$

$$= \frac{1.3.5\ldots(2n-1)}{n!} \times \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} \quad \ldots (4)$$

and so on.

Thus coefficient of z^n in the expansion of (1) is sum of (2), (3) and (4) etc.

$$= \frac{1.3.5\ldots(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)} \cdot x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{n-4} - \ldots\right] = P_n(x)$$

Thus coefficients of z, z^2, z^3 ... etc. in (1) are $P_1(x)$, $P_2(x)$, $P_3(x)$ ...

Hence

$$(1 - 2xz + z^2)^{-1/2} = P_0(x) + zP_1(x) + z^2P_2(x) + z^3P_3(x) + \ldots + z^n P_n(x) + \ldots$$

$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{n=\infty} P_n(x) \cdot z^n.$$

THANK YOU