ORTHOGONALITY OF LAGUERRES FUNCTION

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LAGUERRES POLYNOMIAL

$$\Rightarrow \quad L_n(x) = (-1)^n \left[x^n - \frac{n^2}{1!} x^{n-1} + \frac{n^2 (n-1)^2}{2!} x^{n-2} + \ldots\ldots + (-1)^n n! \right]$$

This is the expression for Laguerre's polynomial.

ORTHOGONAL PROPERTY

Let $\quad f_n(x)=\frac{1}{n!}e^{-x/2}L_n(x)$ ....(1)

$$\int_0^\infty f_m(x)\,f_n(x)\,dx=\int_0^\infty e^{-x}\frac{L_m(x)}{m!}\frac{L_n(x)}{n!}dx=\delta_{m,n}$$

Over the interval $0\le x\le\infty$ 1 when $\delta_{m,n}=\begin{cases}0 \text{ for } m\ne n\\ 1 \text{ for } m=n\end{cases}$

From recurrence relation (4), we know that

$$L_n(x)=e^x\frac{d^n}{dx^n}\left(x^n e^{-x}\right)$$

So, we get

$$\int_0^\infty e^{-x}x^m L_n(x)\,dx=\int_0^\infty e^{-x}x^m e^x\frac{d^n}{dx^n}\left(x^n e^{-x}\right)dx$$

$$=\int_0^\infty x^m\frac{d^n}{dx^n}\left(x^n e^{-x}\right)dx$$

Integrating the R.H.S by parts, we get

$$\int_0^\infty e^{-x}x^m L_n(x)\,dx=\left[x^m\frac{d^{n-1}}{dx^{n-1}}x^n e^{-x}\right]_0^\infty-\int_0^\infty mx^{m-1}\frac{d^{n-1}}{dx^{n-1}}\left(x^n e^{-x}\right)dx$$

$$=(-1)m\int_0^\infty e^{m-1}\frac{d^{n-1}}{dx^{n-1}}\left(x^n e^{-x}\right)dx$$

$$=(-1)^2 m(m-1)\int_0^\infty e^{m-2}\frac{d^{n-2}}{dx^{n-2}}\left(x^n e^{-x}\right)dx$$

$$=\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= (-1)^n m! \int_0^\infty \frac{d^{n-m}}{dx^{n-m}} \left(x^n e^{-x}\right) dx$$

$$= 0 \text{ if } n > m.$$

Replacing n by m, we get $(m < n)$

$$\int_0^\infty e^{-x} x^n L_m dx = 0 \text{ for } m < n$$

$$\Rightarrow \quad \int_0^\infty e^{-x} L_m(x) L_n(x) dx = 0, \text{ if } m \neq n$$

$$\Rightarrow \quad \int_0^\infty e^{-x} \frac{L_m(x)}{m!} \frac{L_n(x)}{n!} dx = 0, \text{ if } m \neq n$$

Taking $m = n$, then

$L_n(x)$ is $(-1)^n x^n$,

$$\therefore \quad \int_0^\infty e^{-x}\{L_n(x)\}^2 dx = (-1)^n \int_0^\infty e^{-x} x^n L_n(x) dx$$

$$= (-1)^n \int_0^\infty e^{-x} x^n e^x \frac{d^n}{dx^n}\left(x^n e^{-x}\right) dx$$

$$= (-1)^n \int_0^\infty x^n \frac{d^n}{dx^n}\left(x^n e^{-x}\right) dx$$

$$= (-1)^{2n} n! \int_0^\infty x^n e^{-x} dx$$

$$= (n!)^2$$

$$\Rightarrow \quad \int_0^\infty e^{-x/2} \frac{L_n(x)}{n!} e^{-x/2} \frac{L_n(x)}{n!} dx = 1 \qquad \text{.........(3)}$$

Combining (2) and (3), we get

$$\int_0^\infty f_m(x) f_n(x) dx = \int_0^\infty e^{-x/2} \frac{L_m(x)}{m!} e^{-x/2} \frac{L_n(x)}{n!} dx = \delta_{m,n}$$